A Common Solution to a Pair of Linear Matrix Equations Over a Principal Ideal Domain

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ABSTRACT

A necessary and sufficient condition for the existence of a common solution to a pair of linear matrix equations over a principal ideal domain is obtained. The equations are of the type $A_i = B_i X C_i$ for $i = 1,2$. The solvability condition is that the equations each have a solution and a bilateral linear matrix equation made up of the matrices A_i, B_i, and C_i has a solution.

1. INTRODUCTION

This paper is concerned with the problem of stating verifiable solvability conditions for the existence of a common solution to the following pair of linear matrix equations:

$$A_1 = B_1 X C_1, \qquad A_2 = B_2 X C_2. \tag{1}$$

In (1), A_i, B_i, and C_i $(i = 1,2)$ are matrices with elements in a given arbitrary principal ideal domain $\mathscr{R}$ over which X is to be determined.

The problem has been motivated by Woude [1, 2] in the context of *noninteracting control by measurement feedback with or without internal stability*, which can be described as follows: Given a composite linear system having two exogenous inputs and two exogenous outputs in addition to a control input and a measurement output, find a dynamic output feedback

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compensator which processes the measurement output to cancel pairwise the effect of exogenous inputs on exogenous outputs. In the exact versions of these problems, the exogenous outputs are required to be independent of the exogenous inputs. In the "almost" versions, the interaction between the exogenous inputs and outputs is required to be arbitrarily small in an $\mathscr{H}_\infty$ sense. The solvability of the almost noninteracting control problem without the constraint of internal stability has been shown in [1, 2] to be equivalent to the existence of a solution to (1) over $\mathbf{R}(s)$, the field of rational functions in the indeterminate s. The solvability of the exact noninteracting control problem without internal stability, on the other hand, is again equivalent to the solvability of (1) over $\mathscr{R}$ = ring of proper rational functions, where A_i, B_i, and C_i $(i = 1,2)$ are transfer matrices of various subsystems. When the constraint of internal stability is imposed, the central solvability conditions for the exact and almost versions of the problem are the solvability of (1) when $\mathscr{R}$ = ring of proper stable rational functions and $\mathscr{R}$ = ring of stable rational functions, respectively (Akar [3]). Here, A_i, B_i, and C_i $(i = 1,2)$ are system matrices over $\mathscr{R}$ associated with various subsystems.

Necessary and sufficient conditions for the solvability of these equations, in case the above matrices have elements in a field $\mathscr{F}$ and X is sought over $\mathscr{F}$, have been obtained by Mitra [4] and Woude [1]. In [4], the solvability conditions are given in terms of generalized inverses of certain matrices. One of the solvability conditions given in [1] is the following:

PROPOSITION 1.1. *There exists a matrix X such that $A_1 = B_1 X C_1$ and $A_2 = B_2 X C_2$ if and only if for $i = 1,2$ one has*

$$\operatorname{rank} B_i = \operatorname{rank}[B_i, A_i], \qquad \operatorname{rank} C_i = \operatorname{rank}\begin{bmatrix} C_i \\ A_i \end{bmatrix},$$

and

$$\operatorname{rank}\begin{bmatrix} B_1 & 0 & 0 \\ B_2 & 0 & 0 \\ 0 & C_1 & C_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_1 & A_1 & 0 \\ B_2 & 0 & -A_2 \\ 0 & C_1 & C_2 \end{bmatrix}.$$

The last condition above can also be shown to be equivalent to the existence of matrices X and Y over $\mathscr{F}$ such that

$$BX + YC = A,$$

where

$$B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C=\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad A=\begin{bmatrix} A_1 & 0 \\ 0 & -A_2 \end{bmatrix}.$$

Our main result is that the equations $A_1 = B_1XC_1$ and $A_2 = B_2XC_2$ have a common solution over $\mathscr{R}$ iff they are separately solvable over $\mathscr{R}$ and a matrix equation of the type $BX + YC = A$ is solvable over $\mathscr{R}$. Both of these two conditions are well known in the control theory literature. The first condition, the separate solvability of $A_i = B_iXC_i$, occurs as the solvability condition for disturbance decoupling problems via dynamic output feedback (Ohm, Howze, and Bhattacharyya [5], Özgüler and Eldem [6]). A characterization of all solutions to $A = BXC$ for the case of some particular principal ideal domains is also given in [7].

The second condition is the solvability of a bilateral matrix equation, and the existence of a solution can be checked by using the fundamental result of Roth [8]. This condition can also be stated in terms of skew-primeness of certain matrices. Such conditions occur as the solvability conditions for various output regulation and/or tracking problems (Wolovich and Ferreira [9], Khargonekar and Özgüler [10]).

2. PRELIMINARIES AND NOTATION

In this section, we give certain terminology and facts concerning matrices over a principal ideal domain. All the facts below, stated without proof, can be found in Mac Duffee [11].

Let $\mathscr{R}$ be a principal ideal domain. Also let $\mathscr{R}^{n\times m}$ denote the set of matrices of size $n\times m$ whose elements belong to $\mathscr{R}$. A matrix $A\in\mathscr{R}^{n\times m}$ is said to have rank l if (i) there is an $l\times l$ nonzero minor of A and (ii) every $(l+1)\times(l+1)$ minor of A is zero. If $n=l$ $(m=l)$, A is said to have *full row rank* (*full column rank*). A matrix $U\in\mathscr{R}^{n\times n}$ is called *unimodular* if and only if there exists $U^{\#}\in\mathscr{R}^{n\times n}$ such that $U^{\#}U=UU^{\#}=I$, or equivalently, U is unimodular iff $\det U$ is a unit in $\mathscr{R}$. A matrix $U\in\mathscr{R}^{n\times m}$ is called *right unimodular* iff there exists $U^{\#}\in\mathscr{R}^{m\times n}$ such that $U^{\#}U=I$. Similarly, $U\in\mathscr{R}^{n\times m}$ is called *left unimodular* iff there exists $U^{\#}\in\mathscr{R}^{n\times n}$ such that $UU^{\#}=I$. Two matrices A and B are called *left associates* if there exists a unimodular matrix U such that $A=UB$. It is well known that a matrix $A\in\mathscr{R}^{n\times m}$ is a left associate of a matrix of the form

$$\begin{bmatrix} G \\ 0 \end{bmatrix},$$

where G is of full row rank (see also Vidyasagar [12]). If three matrices over $\mathcal{R}$ are in the relation $A = CG$, then G is called a *right divisor* of A and A is called a *left multiple* of G. A *greatest right divisor* of A is a right divisor which is a left multiple of every right divisor of A.

Let M be a full column rank matrix in $\mathcal{R}^{n\times m}$. A greatest right divisor of M is a square nonsingular matrix L over $\mathcal{R}$ such that $M = UL$ for a right unimodular U. A *greatest common right divisor* (gcrd) G of two matrices A and B is a greatest right divisor of $\begin{bmatrix} A \\ B \end{bmatrix}$. Every pair of matrices A and B with elements in $\mathcal{R}$ have a gcrd expressible in the form $G = PA + QB$ with P and Q over $\mathcal{R}$. If the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is of full column rank, the matrices A and B have a nonsingular gcrd G. If A and B have a nonsingular gcrd G, every gcrd of A and B is of the form UG where U is unimodular. $A \in \mathcal{R}^{n\times m}$ and $B \in \mathcal{R}^{k\times m}$ are called *right coprime* iff a gcrd of A and B is unimodular. Let U be a unimodular matrix such that

$$U\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix},$$

where $\begin{bmatrix} V \\ 0 \end{bmatrix}$ is the Hermite row form of $\begin{bmatrix} A \\ B \end{bmatrix}$. It follows that $V \in \mathcal{R}^{m\times m}$ is a gcrd of A and B, and is unimodular if and only if A and B are right coprime. Partitioning U and $W := U^{-1}$ appropriately, one can immediately show that

$$\begin{bmatrix} V^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} A & W_{12} \\ B & W_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where $\begin{bmatrix} W_{12} \\ W_{22} \end{bmatrix}$ is composed of the last $n + k - m$ columns of W. A *greatest common left divisor* of two matrices over $\mathcal{R}$, *left coprimeness*, and a *greatest left divisor* of a matrix over $\mathcal{R}$ can be defined similarly or via matrix transposition. Two matrices A and B are called *equivalent* iff there exist unimodular matrices M and N of suitable sizes such that $A = MBN$.

Let A and B be two matrices over $\mathcal{R}$ of sizes $p \times q$ and $q \times r$, respectively. A and B are called *externally skew-prime* if and only if there exist $X \in \mathcal{R}^{q\times p}$ and $Y \in \mathcal{R}^{r\times q}$ satisfying $XA + BY = I$. It is shown in [13] that A and B are externally skew-prime if and only if there exist matrices $\tilde{B}, \tilde{A}$ over $\mathcal{R}$ such that $AB = \tilde{B}\tilde{A}$ with A and $\tilde{B}$ left coprime and B and $\tilde{A}$ right coprime.

We now present two lemmas. Lemma 2.1 concerns the solvability of equations of the type $A = BXC$, and it will be used in establishing our main result. Lemma 2.2 concerns the solvability of the bilateral matrix equation that figures as one of the main solvability conditions for (1).

LEMMA 2.1. *Let* $A \in \mathscr{R}^{p\times q}$, $B \in \mathscr{R}^{p\times r}$, *and* $C \in \mathscr{R}^{s\times q}$. *Also let* $M \in \mathscr{R}^{p\times p}$ *and* $N \in \mathscr{R}^{q\times q}$ *be unimodular matrices such that*

$$MB = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \qquad CN = \begin{bmatrix} \hat{C} & 0 \end{bmatrix}$$

with $\hat{B}$ *of full row rank in* $\mathscr{R}^{k\times r}$ *and* $\hat{C}$ *of full column in* $\mathscr{R}^{s\times l}$, *where* $k := \operatorname{rank} B$ *and* $l := \operatorname{rank} C$. *Set*

$$\hat{A} := MAN = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix},$$

partitioned so that $\hat{A}_{11}$ *is in* $\mathscr{R}^{k\times l}$. *Further, let L be a greatest left divisor of* $\hat{B}$ *and let R be a greatest right divisor of* $\hat{C}$ *so that*

$$\hat{B} = LU, \qquad \hat{C} = VR$$

for a left unimodular U and a right unimodular V. The equation

$$A = BXC$$

has a solution X over $\mathscr{R}^{r\times s}$ *if and only if*

(i) $\hat{A}_{12} = 0$, $\hat{A}_{21} = 0$, $\hat{A}_{22} = 0$,
(ii) $L^{-1}\hat{A}_{11}R^{-1} \in \mathscr{R}^{k\times l}$.

Proof. Let X be in $\mathscr{R}^{r\times s}$ satisfying $A = BXC$. This implies

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} X \begin{bmatrix} \hat{C} & 0 \end{bmatrix},$$

which implies (i). Note that $\hat{A}_{11} = \hat{B}X\hat{C}$, which yields $UXV = L^{-1}\hat{A}_{11}R^{-1}$, where the left hand side is over $\mathscr{R}$. Thus, (ii) holds. Conversely, let $U^{\#} \in \mathscr{R}^{r\times k}$ and $V^{\#} \in \mathscr{R}^{l\times s}$ be such that

$$UU^{\#} = I, \qquad V^{\#}V = I.$$

On setting

$$X := U^{\#} L^{-1} \hat{A}_{11} R^{-1} V^{\#}$$

and by using (i) and (ii), $A = BXC$ holds with $X \in \mathscr{R}^{r \times s}$. ■

LEMMA 2.2. *Let $A \in \mathscr{R}^{l \times m}$, $B \in \mathscr{R}^{l \times p}$, and $C \in \mathscr{R}^{q \times m}$. The following statements are equivalent:*

(i) $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ *and* $\begin{bmatrix} B & A \\ 0 & C \end{bmatrix}$ *are equivalent over $\mathscr{R}$.*
(ii) *The matrix equation $BX + YC = A$ has a solution X and Y over $\mathscr{R}$.*
(iii) $\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}$ *and* $\begin{bmatrix} I & A \\ 0 & C \end{bmatrix}$ *are externally skew-prime.*

Proof. (i) $\Leftrightarrow$ (ii): See Roth [8] and Kučera [14].
The equivalence of (ii) and (iii) is due to Fuhrmann [15].
(iii) $\Rightarrow$ (ii): Suppose (iii) holds. It follows from the definition of skew-primeness that

$$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

On defining $X := X_1 A - X_2$ and $Y = -Y_2$ it is clear that (ii) is satisfied.
(ii) $\Rightarrow$ (iii): Suppose (ii) holds. It is easy to check with

$$X_1 := 0, \qquad X_2 := -X, \qquad X_3 := 0, \qquad X_4 := I$$

and

$$Y_1 := I, \qquad Y_2 := -Y, \qquad Y_3 := 0, \qquad Y_4 := 0$$

that the following equality is satisfied:

$$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

This yields (iii). ■

3. MAIN RESULT

Let $\mathscr{R}$ be a principal ideal domain. Also, let $A_1 \in \mathscr{R}^{p_1 \times q_1}$, $A_2 \in \mathscr{R}^{p_2 \times q_2}$, $B_1 \in \mathscr{R}^{p_1 \times r}$, $B_2 \in \mathscr{R}^{p_2 \times r}$, $C_1 \in \mathscr{R}^{s \times q_1}$, and $C_2 \in \mathscr{R}^{s \times q_2}$. Let $M_1 \in \mathscr{R}^{p_1 \times p_1}$, $M_2 \in \mathscr{R}^{p_2 \times p_2}$, $N_1 \in \mathscr{R}^{q_1 \times q_1}$, and $N_2 \in \mathscr{R}^{q_2 \times q_2}$ be unimodular matrices such that

$$M_1 B_1 = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \qquad M_2 B_2 = \begin{bmatrix} \hat{B}_2 \\ 0 \end{bmatrix},$$

$$C_1 N_1 = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}, \qquad C_2 N_2 = \begin{bmatrix} \hat{C}_2 & 0 \end{bmatrix}, \tag{2}$$

where $\hat{B}_1 \in \mathscr{R}^{k_1 \times r}$, $\hat{B}_2 \in \mathscr{R}^{k_2 \times r}$ are of full row rank and $\hat{C}_1 \in \mathscr{R}^{s \times l_1}$, $\hat{C}_2 \in \mathscr{R}^{s \times l_2}$ are of full column rank. Note that M_1, M_2 can be chosen to be some unimodular matrices yielding Hermite row forms B_1, B_2, and N_1, N_2 can be chosen to be some unimodular matrices yielding Hermite column forms for C_1, C_2, respectively. Set

$$\hat{A}_1 := M_1 A_1 N_1 = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{13} & \hat{A}_{14} \end{bmatrix}, \qquad \hat{A}_2 := M_2 A_2 N_2 = \begin{bmatrix} \hat{A}_{21} & \hat{A}_{22} \\ \hat{A}_{23} & \hat{A}_{24} \end{bmatrix}, \tag{3}$$

partitioned so that $\hat{A}_{11} \in \mathscr{R}^{k_1 \times l_1}$ and $\hat{A}_{21} \in \mathscr{R}^{k_2 \times l_2}$. Further, let L_1, L_2 be greatest left divisors of $\hat{B}_1, \hat{B}_2$, and R_1, R_2 be greatest right divisors of $\hat{C}_1, \hat{C}_2$, respectively, such that

$$\hat{B}_1 = L_1 U_1, \qquad \hat{B}_2 = L_2 U_2, \qquad \hat{C}_1 = V_1 R_1, \qquad \hat{C}_2 = V_2 R_2 \tag{4}$$

for some left unimodular U_1, U_2 and right unimodular V_1, V_2. Set

$$W_1 := L_1^{-1} \hat{A}_{11} R_1^{-1}, \qquad W_2 := L_2^{-1} \hat{A}_{21} R_2^{-1}. \tag{5}$$

Now, we are ready to state the main result of this paper.

THEOREM 3.1. *The linear matrix equations*

$$A_1 = B_1 X C_1, \qquad A_2 = B_2 X C_2$$

have a common solution X over $\mathscr{R}$ if and only if the following conditions *hold*:

(C1) $\hat{A}_{i2}=0$, $\hat{A}_{i3}=0$, $\hat{A}_{i4}=0$ *for* $i=1,2$.

(C2) $W_i \in \mathscr{R}^{k_i \times l_i}$, $i=1,2$.

(C3) *There exist* $X_1 \in \mathscr{R}^{r\times l_1}$, $X_2 \in \mathscr{R}^{r\times l_2}$, $Y_1 \in \mathscr{R}^{k_1\times s}$, *and* $Y_2 \in \mathscr{R}^{k_2\times s}$ *such that*

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & -W_2 \end{bmatrix}. \tag{6}$$

Proof. "Only if": Suppose $X \in \mathscr{R}^{r\times s}$ is such that (1) holds. By Lemma 2.1, this immediately implies (C1) and (C2). It is easy to check that with

$$X_1 := XV_1, \qquad X_2 := 0, \qquad Y_1 := 0, \qquad Y_2 := -U_2 X,$$

the equality (6) also holds.

"If": Suppose (C1), (C2), and (C3) hold. (C1) and (C2) together with Lemma 2.1 imply that there exist Z_1 and Z_2 over $\mathscr{R}$ such that

$$U_1 Z_1 V_1 = W_1, \qquad U_2 Z_2 V_2 = W_2. \tag{7}$$

Let $M \in \mathscr{R}^{r\times r}$ and $N \in \mathscr{R}^{s\times s}$ be unimodular matrices such that

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} M = \begin{bmatrix} \tilde{U}_1 & 0 \\ \tilde{U}_2 & 0 \end{bmatrix}, \qquad N\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

where $\tilde{U}_1 \in \mathscr{R}^{k_1\times t}$, $\tilde{U}_2 \in \mathscr{R}^{k_2\times t}$, $\tilde{V}_1 \in \mathscr{R}^{d\times l_1}$, $\tilde{V}_2 \in \mathscr{R}^{d\times l_2}$, $\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix}$ is of full column rank, and $[\tilde{V}_1 \ \tilde{V}_2]$ is of full row rank. Also note that M and N can be chosen to be some unimodular matrices yielding Hermite column form for the matrix $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ and Hermite row form for the matrix $[V_1 \ V_2]$, respectively. It is clear that, $\tilde{U}_1, \tilde{U}_2$ are left unimodular and $\tilde{V}_1, \tilde{V}_2$ are right unimodular. Now, let

$$M^{-1}Z_1N^{-1} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{13} & Z_{14} \end{bmatrix}, \qquad M^{-1}Z_2N^{-1} = \begin{bmatrix} Z_{21} & Z_{22} \\ Z_{23} & Z_{24} \end{bmatrix},$$

partitioned so that $Z_{11} \in \mathscr{R}^{t\times d}$ and $Z_{21} \in \mathscr{R}^{t\times d}$. By (7), they satisfy

$$\tilde{U}_1 Z_{11} \tilde{V}_1 = W_1, \qquad \tilde{U}_2 Z_{21} \tilde{V}_2 = W_2. \tag{9}$$

Defining

$$M^{-1}[X_1 X_2] = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \\ \tilde{X}_3 & \tilde{X}_4 \end{bmatrix}, \qquad \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} N^{-1} = \begin{bmatrix} \tilde{Y}_1 & \tilde{Y}_3 \\ \tilde{Y}_2 & \tilde{Y}_4 \end{bmatrix}, \tag{10}$$

partitioned so that $\tilde{X}_1 \in \mathscr{R}^{t\times l_1}$, $\tilde{X}_2 \in \mathscr{R}^{t\times l_2}$, $\tilde{Y}_1 \in \mathscr{R}^{k_1\times d}$, and $\tilde{Y}_2 \in \mathscr{R}^{k_2\times d}$, (6) and (10) yield the equality

$$\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \end{bmatrix} + \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & -W_2 \end{bmatrix}. \tag{11}$$

Note that if we can find a common solution $X_s \in \mathscr{R}^{t\times d}$ to

$$\tilde{U}_1 X_s \tilde{V}_1 = W_1, \qquad \tilde{U}_2 X_s \tilde{V}_2 = W_2$$

by using (9) and (11), then the matrix

$$M \begin{bmatrix} X_s & 0 \\ 0 & 0 \end{bmatrix} N$$

will be a common solution to the equations (1). This is clear by (2), (3), (4), (5), and (C1).

Let G be a greatest common right divisor of $\tilde{U}_1$ and $\tilde{U}_2$, so that

$$\tilde{U}_1 = \Theta_1 G, \qquad \tilde{U}_2 = \Theta_2 G \tag{12}$$

for some right coprime Θ_1 and Θ_2 over $\mathscr{R}$. Since $\tilde{U}_1$ and $\tilde{U}_2$ are left unimodular, there exist $\tilde{U}_1^{\#} \in \mathscr{R}^{t\times k_1}$ and $\tilde{U}_2^{\#} \in \mathscr{R}^{t\times k_2}$ such that

$$\tilde{U}_1 \tilde{U}_1^{\#} = I, \qquad \tilde{U}_2 \tilde{U}_2^{\#} = I. \tag{13}$$

Setting $\Theta_1^{\#} := G\tilde{U}_1^{\#}$ and $\Theta_2^{\#} := G\tilde{U}_2^{\#}$ will immediately yield

$$\Theta_1 \Theta_1^{\#} = I, \qquad \Theta_2 \Theta_2^{\#} = I. \tag{14}$$

Further, since Θ_1 and Θ_2 are right coprime, there exist matrices $K_1 \in \mathscr{R}^{t\times k_1}$,

$K_2 \in \mathscr{R}^{t \times k_2}$, $\tilde{K}_1 \in \mathscr{R}^{k_1 \times k_1+k_2-t}$, $\tilde{K}_2 \in \mathscr{R}^{k_2 \times k_1+k_2-t}$, $\tilde{\Theta}_1 \in \mathscr{R}^{k_1+k_2-t \times k_1}$, and $\tilde{\Theta}_2 \in \mathscr{R}^{k_1+k_2-t \times k_2}$ such that the following identity holds:

$$\begin{bmatrix} K_1 & K_2 \\ \tilde{\Theta}_1 & \tilde{\Theta}_2 \end{bmatrix} \begin{bmatrix} \Theta_1 & \tilde{K}_1 \\ \Theta_2 & \tilde{K}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{15}$$

Similarly, let F be a greatest common left divisor of $\tilde{V}_1$ and $\tilde{V}_2$, so that

$$\tilde{V}_1 = F\Psi_1, \qquad \tilde{V}_2 = F\Psi_2 \tag{16}$$

for some left coprime Ψ_1 and Ψ_2 over $\mathscr{R}$. Since $\tilde{V}_1$ and $\tilde{V}_2$ are left unimodular, there exist $\tilde{V}_1^{\#} \in \mathscr{R}^{l_1 \times d}$ and $\tilde{V}_2^{\#} \in \mathscr{R}^{l_2 \times d}$ such that

$$\tilde{V}_1^{\#}\tilde{V}_1 = I, \qquad \tilde{V}_2^{\#}\tilde{V}_2 = I. \tag{17}$$

Setting $\Psi_1^{\#} := \tilde{V}_1^{\#}F$ and $\Psi_2^{\#} := \tilde{V}_2^{\#}F$, we obtain

$$\Psi_1^{\#}\Psi_1 = I, \qquad \Psi_2^{\#}\Psi_2 = I. \tag{18}$$

Moreover, since Ψ_1 and Ψ_2 are left coprime, there exist matrices $L_1 \in \mathscr{R}^{l_1 \times d}$, $L_2 \in \mathscr{R}^{l_2 \times d}$, $\tilde{L}_1 \in \mathscr{R}^{l_1+l_2-d \times l_1}$, $\tilde{L}_2 \in \mathscr{R}^{l_1+l_2-d \times l_2}$, $\tilde{\Psi}_1 \in \mathscr{R}^{l_1 \times l_1+l_2-d}$, and $\tilde{\Psi}_2 \in \mathscr{R}^{l_2 \times l_1+l_2-d}$ such that the following identity holds:

$$\begin{bmatrix} \Psi_1 & \Psi_2 \\ \tilde{L}_1 & \tilde{L}_2 \end{bmatrix} \begin{bmatrix} L_1 & \tilde{\Psi}_1 \\ L_2 & \tilde{\Psi}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{19}$$

Now we define the following matrices which we will soon make use of:

$$\tilde{X} := \tilde{X}_1 L_1 + \tilde{X}_2 L_2, \tag{20}$$

$$\tilde{Y} := K_1\tilde{Y}_1 + K_2\tilde{Y}_2, \tag{21}$$

$$\hat{X} := \tilde{X}_1\tilde{\Psi}_1 + \tilde{X}_2\tilde{\Psi}_2, \tag{22}$$

$$\hat{Y} := \tilde{\Theta}_1\tilde{Y}_1 + \tilde{\Theta}_2\tilde{Y}_2, \tag{23}$$

$$Z := Z_{11} - Z_{21}. \tag{24}$$

On premultiplying equality (11) by

$$\begin{bmatrix} K_1 & K_2 \\ \tilde{\Theta}_1 & \tilde{\Theta}_2 \end{bmatrix},$$

postmultiplying it by

$$\begin{bmatrix} L_1 & \tilde{\Psi}_1 \\ L_2 & \tilde{\Psi}_2 \end{bmatrix},$$

and using (20), (21), (22), (23), and (24), we obtain the following equalities:

$$G\tilde{X} + \tilde{Y}F = K_1\Theta_1 GZ_{11}F\Psi_1 L_1 - K_2\Theta_2 GZ_{21}F\Psi_2 L_2, \tag{25}$$

$$G\hat{X} = K_1\Theta_1 GZ_{11}F\Psi_1\tilde{\Psi}_1 - K_2\Theta_2 GZ_{21}F\Psi_2\tilde{\Psi}_2, \tag{26}$$

$$\hat{Y}F = \tilde{\Theta}_1\Theta_1 GZ_{11}F\Psi_1 L_1 - \tilde{\Theta}_2\Theta_2 GZ_{21}F\Psi_2 L_2, \tag{27}$$

$$0 = \tilde{\Theta}_1\Theta_1 GZF\Psi_1\tilde{\Psi}_1 = \tilde{\Theta}_2\Theta_2 GZF\Psi_2\tilde{\Psi}_2. \tag{28}$$

We now show that the matrix

$$X := M\begin{bmatrix} X_s & 0 \\ 0 & 0 \end{bmatrix}N,$$

extended so that $X \in \mathscr{R}^{r\times s}$ and where

$$\begin{aligned} X_s = Z_{11} &+ \left(Z_{11}F\Psi_1\tilde{\Psi}_1 - \hat{X}\right)\left(\tilde{L}_2 - \tilde{L}_1\Psi_1^{\#}\Psi_2\right)\tilde{V}_2^{\#} \\ &+ \tilde{U}_2^{\#}\left(\tilde{K}_2 - \Theta_2\Theta_1^{\#}\tilde{K}_1\right)\left(\tilde{\Theta}_1\Theta_1 GZ_{11} - \hat{Y}\right) \\ &+ \left(\tilde{U}_1^{\#}\tilde{U}_1 - I\right)\left(\tilde{X} - Z_{11}F\Psi_1 L_1\right)\left(I - \Psi_1\Psi_1^{\#}\right)\Psi_2\tilde{V}_2^{\#} \\ &+ \tilde{U}_2^{\#}\Theta_2\left(\Theta_1^{\#}\Theta_1 - I\right)\left(\tilde{Y} + K_2\Theta_2 GZ_{11}\right)\left(I - \tilde{V}_1\tilde{V}_1^{\#}\right), \end{aligned} \tag{29}$$

is a common solution to the equations (1). We claim that X_s is a common solution to the following equations:

$$\tilde{U}_1 X_s \tilde{V}_1 = W_1, \tag{30}$$

$$\tilde{U}_2 X_s \tilde{V}_2 = W_2. \tag{31}$$

In order for (30) to be satisfied, the last four terms of X_s, when premultiplied by $\tilde{U}_1$ and postmultiplied by $\tilde{V}_1$, should vanish. By (12) and (16), $\tilde{U}_1 X_s \tilde{V}_1 = \Theta_1 G X_s F \Psi_1$. Note that

$$\begin{aligned}\Theta_1 G\left(Z_{11} F \Psi_1 \tilde{\Psi}_1 - \hat{X}\right) &= \Theta_1 K_2 \Theta_2 G Z F \Psi_1 \tilde{\Psi}_1 \\ &= \tilde{K}_1 \tilde{\Theta}_2 \Theta_2 G Z F \Psi_2 \tilde{\Psi}_2 \\ &= 0\end{aligned}$$

by (26), (15), and (28) respectively. Also

$$\begin{aligned}\left(\tilde{\Theta}_1 \Theta_1 G Z_{11} - \hat{Y}\right) F \Psi_1 &= \tilde{\Theta}_1 \Theta_1 G Z F \Psi_2 L_2 \Psi_1 \\ &= \tilde{\Theta}_1 \Theta_1 G Z F \Psi_1 \tilde{\Psi}_1 \tilde{L}_1 \\ &= 0\end{aligned}$$

by (27), (19), and (28) respectively.

Finally, $\tilde{U}_1(\tilde{U}_1^{\#}\tilde{U}_1 - I) = 0$ by (13) and $(I - \tilde{V}_1 \tilde{V}_1^{\#})\tilde{V}_1 = 0$ by (17), implying that $\tilde{U}_1 X_s \tilde{V}_1 = \tilde{U}_1 Z_{11} \tilde{V}_1 = W_1$ by (9). Therefore, X_s defined as in (29) satisfies (30).

Now, consider (31). Note that, in order for (31) to be satisfied, in view of (9) and (24), the last four terms, when premultiplied by $\tilde{U}_2$ and postmultiplied by $\tilde{V}_2$, should be equal to $-\tilde{U}_2 Z \tilde{V}_2$. By (12) and (16), $\tilde{U}_2 X_s \tilde{V}_2 = \Theta_2 G X_s F \Psi_2$. Also, note that

$$\begin{aligned}&\tilde{U}_2\left(Z_{11} F \Psi_1 \tilde{\Psi}_1 - \hat{X}\right)\left(\tilde{L}_2 - \tilde{L}_1 \Psi_1^{\#} \Psi_2\right) \tilde{V}_2^{\#} \tilde{V}_2 \\ &\quad = \Theta_2\left(K_2 \Theta_2 G Z F \Psi_1 \tilde{\Psi}_1\right)\left(\tilde{L}_2 - \tilde{L}_1 \Psi_1^{\#} \Psi_2\right) \\ &\quad = \Theta_2\left(-K_2 \Theta_2 G Z F \Psi_1 L_1 - K_2 \Theta_2 G Z F \Psi_1 \Psi_1^{\#} + K_2 \Theta_2 G Z F \Psi_1 L_1 \Psi_1 \Psi_1^{\#}\right) \Psi_2\end{aligned}$$

by (12), (17), (26), and (19) respectively. Also

$$\begin{aligned}
&\tilde{U}_2\tilde{U}_2^{\#}\left(\tilde{K}_2 - \Theta_2\Theta_1^{\#}\tilde{K}_1\right)(\tilde{\Theta}_1\Theta_1 GZ_{11} - \hat{Y})\tilde{V}_2 \\
&\quad = \left(\tilde{K}_2 - \Theta_2\Theta_1^{\#}\tilde{K}_1\right)(\tilde{\Theta}_1\Theta_1 GZ_{11} - \hat{Y})F\Psi_2 \\
&\quad = \left(\tilde{K}_2 - \Theta_2\Theta_1^{\#}\tilde{K}_1\right)(\tilde{\Theta}_1\Theta_1 GZF\Psi_2 L_2)\Psi_2 \\
&\quad = \Theta_2\left(-K_1\Theta_1 GZF\Psi_2 L_2 - \Theta_1^{\#}\Theta_1 GZF\Psi_2 L_2 + \Theta_1^{\#}\Theta_1 K_1\Theta_1 GZF\Psi_2 L_2\right)\Psi_2
\end{aligned}$$

by (13), (16), (27), and (15). Moreover,

$$\begin{aligned}
&\tilde{U}_2\left(\tilde{U}_1^{\#}\tilde{U}_1 - I\right)(\tilde{X} - Z_{11}F\Psi_1 L_1)\left(I - \Psi_1\Psi_1^{\#}\right)\Psi_2\tilde{V}_2^{\#}\tilde{V}_2 \\
&\qquad + \tilde{U}_2\tilde{U}_2^{\#}\Theta_2\left(\Theta_1^{\#}\Theta_1 - I\right)(\tilde{Y} + K_2\Theta_2 GZ_{11})\left(I - \tilde{V}_1\tilde{V}_1^{\#}\right)\tilde{V}_2 \\
&\quad = \Theta_2\Big\{\left(\Theta_1^{\#}\Theta_1 - I\right)\big[G(\tilde{X} - Z_{11}F\Psi_1 L_1) \\
&\qquad\qquad + (\tilde{Y} + K_2\Theta_2 GZ_{11})F\big]\left(I - \Psi_1\Psi_1^{\#}\right)\Big\}\Psi_2 \\
&\quad = \Theta_2\left[\left(\Theta_1^{\#}\Theta_1 - I\right)(K_2\Theta_2 GZF\Psi_2 L_2)\left(I - \Psi_1\Psi_1^{\#}\right)\right]\Psi_2
\end{aligned}$$

by (13), (14), (17), (18), (25), (15), and (19) respectively. Finally, by employing (15), (19), and (28) several times, we obtain $\tilde{U}_2 X_s\tilde{V}_2 = \tilde{U}_2(Z_{11} - Z)\tilde{V}_2 = \tilde{U}_2 Z_{21}\tilde{V}_2$, which also equals W_2 by (9). Therefore, (31) is also satisfied.

Considering the matrix equalities in (8), it is clear that X is a common solution to the following matrix equations:

$$U_1XV_1 = W_1, \qquad U_2XV_2 = W_2. \tag{32}$$

On using (2), (3), (4), and (5) together with (C1), Equation (32) immediately implies X is a common solution to the equations (1). ■

REMARK 3.1. Considering the result of Woude [1] for the case where $\mathscr{R}$ is a field, it is natural to expect that (C3) might be replaced by (C4) below:

(C4) *There exist* X_1, X_2, Y_1, *and* Y_2 *such that*

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2 \end{bmatrix}.$$

Evidently, (C3) implies (C4), which together with (C1) and (C2) is necessary for the solvability of the problem. To show that (C1), (C2), and (C4) are not in general sufficient for the solvability of the problem, let $\mathscr{R} = \mathbf{R}[z]$, the ring of polynomials in the indeterminate z. Let

$$A_1 := z, \qquad A_2 := z, \qquad B_1 := [1 \quad z], \qquad B_2 := [0.5 \quad 0],$$
$$C_1 := \begin{bmatrix} z \\ 0 \end{bmatrix}, \quad \text{and} \quad C_2 := \begin{bmatrix} z \\ 0 \end{bmatrix}.$$

Note that, with

$$Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

$A_1 = B_1 Z_1 C_1$ and $A_2 = B_2 Z_2 C_2$. Therefore, (C1) and (C2) are satisfied. Also, by letting

$$X_1 := \begin{bmatrix} 2z \\ -1 \end{bmatrix}, \qquad X_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad Y_1 := [0 \quad 0], \quad \text{and} \quad Y_2 := [-1 \quad 0],$$

(C4) is also satisfied. Now, suppose that there exists a common solution $X \in \mathbf{R}[z]^{2\times 2}$ to the matrix equations (1), in the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

By simple manipulations, the unique X_{21} is found to be $-1/z$, which is indeed not a polynomial in z. This contradicts $X \in \mathbf{R}[z]^{2\times 2}$.

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